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The relationship between causality, in a classical relativistic field theory, and Lorentz invariance, in the corresponding quantum theory, for a massive spin-one field in external potentials

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Abstract. For reasons of calculational simplicity, a massive spin-one field, interacting with external potentials, is described in terms of the Stückelberg formalism. Some points, concerning necessary and sufficient conditions for the classical theory to be causal, in the sense of Velo and Zwanziger, are clarified. It is then shown that with a simple restriction on the type of interaction considered, the classical theory is causal, in the above sense, if and only if the corresponding quantum theory is Lorentz invariant, in that the S operator in the interaction picture is normal independent. Incidental in this demonstration is the result that the characteristic and Lee–Yang determinants share the same form.

It is indicated how this result generalizes to interactions between Stückelberg, Klein–Gordon and electromagnetic fields; and the validity of the result, in the more usual vector formalism, is established.

1. Introduction

In an earlier paper (Jenkins 1973a), it was shown for a massive spin-one vector field in an external potential, the source being linear and non-derivative in the vector field, that the classical theory is causal, in the sense of Velo and Zwanziger (1969a, b, 1971), if and only if the corresponding quantum field theory, in which the lagrangian is symmetrized in the vector field, is Lorentz invariant, in the sense that the S operator in the interaction picture is normal independent.

Such a relationship, between the classical and quantum problems, is, in the light of the work of Capri (1969) (Schroer *et al* 1970), not especially surprising. For, in that work, it is shown, for linear non-derivative sources, that the latter problem is directly reducible to the former. However, some further particular examples, discussed by Jenkins (1973a), suggest that the above relationship, between causality and Lorentz invariance, is of greater generality.

In the present paper, this relationship is shown to be valid for interactions which may depend, in any manner, on the vector field, its antisymmetric first derivatives and external potentials, provided only that this dependence is such that the lagrangian is quadratic in the time derivatives of the vector field, $V_\mu(x)$, and its zeroth component, $V_0(x)$.

This purpose is effected by a use of the (equivalent) Stückelberg formalism (Stückelberg 1938). The reason for this is that, in the Stückelberg formalism, the field equations are true equations of motion, a fact that will be seen to facilitate later calculations; whereas, in the vector formalism, the field equations contain and imply constraints, these being used to derive the true equations of motion (Velo and Zwanziger 1969a, b).

The plan of the paper is as follows. In § 2, a resume of the approach of Velo and Zwanziger (1969a, b, 1971) to the determination of the causal nature of classical relativistic field theories, is given. Some points in earlier work, concerning necessary and sufficient conditions for a causal theory, are clarified. In § 3, the Lorentz invariance of quantum theories of the Stückelberg field in external potentials is discussed in terms of the Lee–Yang theorem (Lee and Yang 1962) and later variants, which take into account the problems of operator ordering. Section 4 is devoted to a demonstration that the Lee–Yang determinant, of § 3, is essentially the same as the characteristic determinant, of § 2, and the consequent relationship between causality and Lorentz invariance is established. The results are discussed in § 5, and there are two appendixes. In appendix 1 the connection between the vector and Stückelberg formalisms is discussed; whilst in appendix 2, a lemma needed in § 2 is proved.

Throughout this paper, the metric used is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\partial_\mu = \partial/\partial x^\mu$ and the usual summation convention for greek indices is adhered to.

2. Causality

Consider the following lagrangian density for a massive spin-one classical field in the Stückelberg formalism (Stückelberg 1938)

$$\mathscr{L}(x) = \mathscr{L}_0(x) + g\mathscr{L}_\mathrm{I}(x) \tag{1}$$

where the free part is given by

$$\mathscr{L}_0(x) = -\tfrac{1}{2}\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) + \tfrac{1}{2}m^2 A_\mu(x)A^\mu(x) + \tfrac{1}{2}\partial_\mu\theta(x)\partial^\mu\theta(x) - \tfrac{1}{2}m^2\theta^2(x) \tag{2}$$

and the interaction is of the form

$$\mathscr{L}_\mathrm{I}(x) = \mathscr{L}_\mathrm{I}\left(A_\mu(x) + \frac{1}{m}\partial_\mu\theta(x),\ \partial_\lambda A_\rho(x) - \partial_\rho A_\lambda(x),\ \text{external potentials}\right). \tag{3}$$

Note that the major coupling constant g has been extracted from $\mathscr{L}_\mathrm{I}(x)$. The field equations which follow from (1) are

$$(\partial^2 + m^2)A_\mu(x) = -g\frac{\partial\mathscr{L}_\mathrm{I}(x)}{\partial A^\mu} + \partial^\rho\left(g\frac{\partial\mathscr{L}_\mathrm{I}(x)}{\partial\partial^\rho A^\mu}\right) \tag{4}$$

$$(\partial^2 + m^2)\theta(x) = -\partial^\rho\left(g\frac{\partial\mathscr{L}_\mathrm{I}(x)}{\partial\partial^\rho\theta}\right) = -\frac{1}{m}\partial^\rho\left(g\frac{\partial\mathscr{L}_\mathrm{I}(x)}{\partial A^\rho}\right) \tag{5}$$

the last equality following from the form (3) of $\mathscr{L}_\mathrm{I}(x)$. Note that, as a consequence of (4), (5) and the form of (3), the following equation is satisfied:

$$(\partial^2 + m^2)(\partial^\mu A_\mu(x) - m\theta(x)) = 0,$$

a necessary condition for the equivalence of the Stückelberg and usual vector formalisms.

It is readily seen that (4) and (5) form a second-order system of partial differential equations, without constraints, and in which the second time derivatives of $A_\mu(x)$ and $\theta(x)$ all appear. Thus (4) and (5) are true equations of motion. The causal nature of the propagation of the solution of such true equations of motion is determined by their characteristic determinant (Velo and Zwanziger 1969a, b, 1971). On making the replacement $\partial_\mu \to n_\mu$ in the second-order derivative terms of (4) and (5), their characteristic determinant, $D(n)$, is calculated as the determinant of the coefficients of $A_\mu(x)$ and $\theta(x)$ in those terms.

The solutions, n_μ, of the equation $D(n) = 0$ give the normals to the characteristic cones corresponding to (4) and (5). If, for all such n_μ, n_0 is real for any $\boldsymbol{n}$, then the system of equations (4) and (5) is hyperbolic and propagation occurs; if otherwise, the solution of (4) and (5) does not propagate. It now remains to discuss the nature of the propagation in the former case. There are, *a priori*, three possibilities, which are discussed separately below.

(i) At least one solution n_μ is timelike for some values of the major coupling constant. For these values of g, the corresponding characteristic cone lies outside the light cone. Thus propagation can occur across the light cone, and the theory given by (1) is then said to be acausal. Note that this definition of an acausal theory only requires that the propagation be acausal for some, not necessarily all, values of the major coupling constant.

(ii) All the solutions n_μ are null. In this case, the characteristic cones are all the light cone, across which propagation cannot occur; and hence the theory given by (1) is causal.

(iii) At least one solution n_μ is spacelike for some values of the major coupling constant. This case, however, is contained in (i), a point which has been previously overlooked. For, as is shown in the lemma of appendix 2, in this case, there exist other values of g for which a timelike solution exists. Hence the theory, given by (1), is acausal in the sense of the definition in (i).

In the light of (i), (ii) and (iii), it follows that a classical theory is causal, in the sense of Velo and Zwanziger (1969a, b, 1971), if and only if all the solutions of $D(n) = 0$ are null. This is readily seen to be equivalent to $D(n)$ having the form

$$D(n) = (n^2)^5 F \tag{6}$$

where F is a Lorentz invariant functional of $A_\mu(x)$, $\theta(x)$ and the external potentials.

3. Lorentz invariance

For the quantum field theory corresponding to the lagrangian density (1), the S operator in the interaction picture is given by

$$S = T\exp\left(-\mathrm{i}\int_{-\infty}^{\infty} \mathscr{H}_\mathrm{I}(x)\,\mathrm{d}^4x\right) \tag{7}$$

where $\mathscr{H}_\mathrm{I}(x)$ is the interaction hamiltonian, in the interaction picture, which corresponds to $\mathscr{L}(x)$. Now, as is well known, the time-ordered product of two fields contains, in general, non-covariant terms. It is desirable to rewrite (7) in terms of an effective interaction hamiltonian density, $\mathscr{H}_\mathrm{I}^{\mathrm{eff}}(x)$, which is to be used in conjunction with the T^* product, this being obtained from the T product by discarding the non-covariant terms in the latter. In this way, all the effects, which could lead to a possible breakdown of the Lorentz invariance of the S operator, are lumped together in $\mathscr{H}_\mathrm{I}^{\mathrm{eff}}(x)$. The prescription for doing this is given by the Lee–Yang theorem (Lee and Yang 1962), and involves writing

$$S = T^* \exp\left(-\mathrm{i}\int_{-\infty}^{\infty} \mathscr{H}_\mathrm{I}^{\mathrm{eff}}(x)\,\mathrm{d}^4x\right) \tag{8}$$

where

$$\mathscr{H}_{\mathrm{I}}^{\mathrm{eff}}(x) = -g\{\mathscr{L}_{\mathrm{I}}(x)\} + \delta\mathscr{H}(x) + \delta^1\mathscr{H}(x) \tag{9}$$

with { } denoting symmetrization, $\delta\mathscr{H}(x)$ given by

$$\delta\mathscr{H}(x) = \frac{\mathrm{i}}{2}\delta^4(0)\ln\mathscr{D} \tag{10}$$

$\mathscr{D}$ being the determinant of the symmetric matrix of the coefficients of the part of $\mathscr{L}(x)$ quadratic in $\sqrt{\frac{1}{2}}\partial_0 A_\mu(x)$ and $\sqrt{\frac{1}{2}}\partial_0\theta(x)$, and where $\delta^1\mathscr{H}(x)$ is a term, proportional to $(\delta^3(0))^2$, which arises from the symmetrization of $\mathscr{L}_{\mathrm{I}}(x)$.

In their original paper, Lee and Yang (1962) neglected all problems concerned with operator ordering, and hence did neither effect the symmetrization nor include $\delta^1\mathscr{H}(x)$ in (9). These modifications of the original discussion have been discussed recently by several authors including Dowker and Mayes (1971), Suzuki and Hattori (1972) and Kvitky and Mouton (1972). Charap (1973) is also a good source of references. In these discussions, it is noted that $\delta^1\mathscr{H}(x)$ cancels certain unwanted contributions from $-g\{\mathscr{L}_{\mathrm{I}}(x)\}$. Hence the only term in which a breakdown of the Lorentz invariance of the S operator can occur is $\delta\mathscr{H}(x)$. Thus the Lorentz invariance, or not, of the quantum theory corresponding to $\mathscr{L}(x)$, is wholly determined by the Lee–Yang determinant $\mathscr{D}$.

Before proceeding, it should be noted that $\mathscr{D}$, calculated in the manner described above, is given in a frame of reference, in which quantization is on a flat spacelike surface with unit normal $\eta_\mu = (1, 0, 0, 0)$. The result can be generalized to an arbitrary spacelike surface, with unit normal η_μ, by the appropriate insertion of η_μ into $\mathscr{D}$, giving $\mathscr{D}(\eta)$. If $\mathscr{D}(\eta)$ is independent of η, the S operator is Lorentz invariant. If further $\delta^1\mathscr{H}(x) = 0$ and $\mathscr{D}(\eta) = 1$, whence $\delta\mathscr{H}(x) = 0$, the generalized Matthews' rule is said to hold (Matthews 1949, Takahashi 1969). However, if $\mathscr{D}(\eta)$ is explicitly dependent on η_μ, then the S operator is not Lorentz invariant.

On noting that η_μ is a unit vector, ie $\eta^2 = 1$, it follows that the quantum field theory corresponding to $\mathscr{L}(x)$ is Lorentz invariant, in the sense that the S operator is Lorentz invariant, if and only if $\mathscr{D}(\eta)$ has the form

$$\mathscr{D}(\eta) = (\eta^2)^5\mathscr{F} \tag{11}$$

where $\mathscr{F}$ is a Lorentz invariant functional of $A_\mu(x)$, $\theta(x)$ and the external potentials.

Finally, it should be noted that, for the applicability of the Lee–Yang theorem, it is necessary that $\mathscr{L}(x)$ be quadratic in the time derivatives of $A_\mu(x)$ and $\theta(x)$, a restriction which will be assumed satisfied throughout the remainder of this paper.

4. Causality and Lorentz invariance

In this section, the connection between causality and Lorentz invariance is established by demonstrating that $D(n)$ and $\mathscr{D}(\eta)$ have the same form. To this end, the expression for $\mathscr{L}(x)$ is firstly made more explicit. The most general expression, satisfying the requirements (1), (2), (3) and the restriction stated at the end of § 3, may be written in the following form:

$$\mathscr{L}(x) = \mathscr{L}_0(x) + \tfrac{1}{2}A_{kj}(\partial_k A_0(x) - \partial_0 A_k(x))(\partial_j A_0(x) - \partial_0 A_j(x))$$
$$+ B_k\left(A_0(x) + \frac{1}{m}\partial_0\theta(x)\right)(\partial_k A_0(x) - \partial_0 A_k(x)) + \tfrac{1}{2}C\left(A_0(x) + \frac{1}{m}\partial_0\theta(x)\right)^2 + D \tag{12}$$

where A_{kj}, B_k, C depend only on $A_k(x)+m^{-1}\partial_k\theta(x)$, $\partial_k A_j(x)-\partial_j A_k(x)$ and the external potentials, whilst, in addition to this dependence, D depends linearly on $A_0(x)+m^{-1}\partial_0\theta(x)$ and $\partial_k A_0(x)-\partial_0 A_k(x)$, and where repeated latin indices are assumed summed over the values 1, 2, 3. The lack of manifest Lorentz invariance in (12) will be convenient for later calculations.

The field equations (3) and (4) may now be written in the form

$$-\partial_0^2 A_0(x) = f_0 \tag{13}$$

$$\partial_0^2 A_k(x) = -\tfrac{1}{2}(A_{ki}+A_{ik})\partial_0^2 A_i(x)+\frac{1}{m}B_k\partial_0^2\theta(x)+f_k \tag{14}$$

$$\partial_0^2\theta(x) = -\frac{C}{m^2}\partial_0^2\theta(x)+\frac{B_k}{m}\partial_0^2 A_k(x)+h \tag{15}$$

where f_0, f_k and h depend on $A_\mu(x)$, $\theta(x)$, the first and second space derivatives and the first time derivatives of these fields and the external potentials. With the field equations written in the above form, it is easy to read off the characteristic determinant in a frame in which $n_\mu = (n_0, 0, 0, 0)$ and the result is

$$D(n) = (n_0^2)^5 \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & \delta_{ik}+\tfrac{1}{2}(A_{ik}+A_{ki}) & & -\dfrac{B_i}{m} \\ 0 & & & & \\ 0 & & -\dfrac{B_k}{m} & & 1+\dfrac{C}{m^2} \end{vmatrix} . \tag{16}$$

Next, the Lee–Yang determinant, $\mathscr{D}(\eta)$, is read off from (12). On remembering that $\eta^2 = 1$, it is, in a frame in which $\eta_\mu = (1, 0, 0, 0)$, readily seen to be

$$\mathscr{D}(\eta) = D(\eta)$$

with $D(\eta)$ given by (16). Thus the connection between the characteristic and Lee–Yang determinants is established.

On account of this result, and the necessary and sufficient conditions for the causality and Lorentz invariance of the classical and quantum theories of the Stückelberg field, discussed in §§ 2 and 3, respectively, it follows that a classical theory of the Stückelberg field interacting with external potentials is causal, in the sense of Velo and Zwanziger (1969a, b, 1971), if and only if the corresponding quantum field theory is Lorentz invariant, in that the S operator in the interaction picture is normal independent; provided only that the interaction is of the form (3) and satisfies the restriction of being quadratic in the time derivatives of the Stückelberg field components.

5. Discussion

In their work on acausal propagation, Velo and Zwanziger (1971) remark that, for causal behaviour, the characteristic cone must lie inside or on the light cone, whilst acausal behaviour will occur if it lies outside the light cone. In the light of the results of § 2 and appendix 2, this remark must, for the Stückelberg field interacting with external potentials, be clarified. For then the former case above, contains the possibility (iii), which involves the occurrence of acausal propagation for some values of the major coupling constant. Thus it is perhaps more illuminating to say that a theory of the Stückelberg field interacting with external potentials is satisfactory, from the standpoint of causal propagation, if and only if the characteristic cones are all the light cone. This view is consistent with a remark of Schroer *et al* (1970), concerning the particular case of sources linear in the field, when the quantum problem is directly reducible to the classical problem (Capri 1969). They remark that, if the characteristic cone lies inside the light cone, then in the quantum theory, the field commutation relations vanish outside a cone contained in the light cone, and that this result is not consistent with the conventional concept of microcausality.

The main result of this paper is that stated in the last paragraph of the previous section, for the Stückelberg field interacting with external potentials†. It should be noted that the essential property of the Stückelberg formalism, which allowed such a simple derivation of this result, is that of the field equations being true equations of motion. This being so, it is easy to see that the arguments of this paper may be readily extended to include, for example, interaction between Stückelberg, Klein–Gordon and electromagnetic fields, and external potentials. In this more general case, the relationship between causality and Lorentz invariance remains valid, provided that the gauge-invariant lagrangian density is quadratic in the time derivatives of all the fields, and that its interaction part has the form

$$\mathscr{L}_{\mathrm{I}}(x) = \mathscr{L}_{\mathrm{I}}\left(A_\mu(x)+\frac{1}{m}\partial_\mu\theta(x), \partial_\lambda A_\rho(x)-\partial_\rho A_\lambda(x), \Phi(x), \partial_\nu\Phi(x), \mathscr{A}_\gamma(x), F_{\alpha\beta}(x), \begin{matrix}\text{external}\\ \text{potentials}\end{matrix}\right)$$

with $\Phi(x)$ the Klein–Gordon field, and $\mathscr{A}_\gamma(x)$ and $F_{\alpha\beta}(x)$ the electromagnetic potential and field-strength tensor, respectively. It should be noted that, incidental in the main result and its above generalization, is the result that the characteristic and Lee–Yang determinants share the same form.

Although the Stückelberg formalism has been used throughout this paper, the results are also valid in the more usual vector formalism. The connection between these formalisms and the relation between the above results therein are discussed in appendix 1. However, one point is worth noting here, and that is that the restriction of the dependence of $\mathscr{L}_{\mathrm{I}}(x)$ to being quadratic in the time derivatives of the field components, in the Stückelberg formalism, becomes, in the vector formalism, a restriction of $\mathscr{L}_{\mathrm{I}}(x)$ to being quadratic in $V_0(x)$ and $\partial_0 V_k(x)-\partial_k V_0(x)$.

Finally, it is remarked that the above generalization of the result of this paper is consistent with the conclusions of Jenkins (1973b), concerning both the electromagnetic interaction of a massive spin-one vector field with arbitrary magnetic dipole moment and the massive Yang–Mills field.

† Recently, Mathews and Seetharaman (1973) have discussed further the causal nature of the theory of a massive spin-one vector field in a constant, uniform antisymmetric second-rank tensor external potential. Although the characteristic cones are all the light cone (Velo and Zwanziger 1969b), Mathews and Seetharaman find that the field equations possess a tachyonic solution, and consequently claim, contrary to the result of Velo and Zwanziger (1969b), that the theory is acausal. In this example, there can be no acceleration across the light cone; hence any acausality must appear as an inherent acausality in the nature of tachyons. However, whether or not there is any inherent acausality in the nature of tachyons is a problem, as yet unresolved, although the latter view seems more popular at present. In the light of these remarks, it must be emphasized that the main result of the present paper involves causality in the sense of Velo and Zwanziger (1969a, b, 1971), and that such a simple result would not hold if causality were defined following Mathews and Seetharaman (1973).

Appendix 1

Consider the following lagrangian density for a massive spin-one vector field interacting with external potentials:

$$\mathscr{L}(x) = -\tfrac{1}{4}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))(\partial^\mu V^\nu(x) - \partial^\nu V^\mu(x)) + \tfrac{1}{2}m^2 V_\mu(x)V^\mu(x)$$
$$+g\mathscr{L}_I(V_\mu(x), \partial_\lambda V_\rho(x) - \partial_\rho V_\lambda(x), \text{external potentials}). \tag{A.1}$$

The field equations are

$$L^\mu \equiv (\partial^2 + m^2)V_\mu(x) - \partial_\mu \partial^\lambda V_\lambda(x) + g\frac{\partial \mathscr{L}_I}{\partial V^\mu}(x) - g\partial^\lambda\left(\frac{\partial \mathscr{L}_I}{\partial \partial^\lambda V^\mu}(x)\right) = 0 \tag{A.2}$$

with $L_0 = 0$ being a primary constraint.

The Stückelberg split (Stückelberg 1938) of the field $V_\mu(x)$ is effected by writing

$$V_\mu(x) = A_\mu(x) + \frac{1}{m}\partial_\mu \theta(x) \tag{A.3}$$

with $A_\mu(x)$ and $\theta(x)$ being unique up to a gauge transformation,

$$A_\mu(x) \to A_\mu(x) + \frac{1}{m}\partial_\mu \Lambda(x)$$

$$\theta(x) \to \theta(x) - \Lambda(x),$$

with $\Lambda(x)$ satisfying the Klein–Gordon equation, and subject to the constraints

$$\mathscr{C} \equiv \partial^\mu A_\mu - m\theta = 0 \tag{A.4}$$

and $L_0 = 0$, with $V_\mu(x)$ written as in (A.3). These constraints and the gauge invariance suffice to reduce the number of independent field components from five to the required three. The gauge-invariant lagrangian density, in the Stückelberg formalism, corresponding to (A.1) is then just (1); whilst the consequent field equations are (4) and (5). These field equations are supplemented by the constraints $L_0 = 0$ and $\mathscr{C} = 0$, assumed valid at $t = 0$. For consistency, (4) and (5) must preserve these constraints in time.

To see this, firstly note that, on using (A.3) in (A.2), (4) may be rewritten as

$$L_\mu + \partial_\mu \mathscr{C} = 0. \tag{A.5}$$

Now, since $L_0 = \mathscr{C} = 0$ at $t = 0$ and, as follows from (A.2) and (A.4), $\partial^\mu L_\mu = m^2\mathscr{C}$, it follows that $\partial_0 \mathscr{C} = 0$ at $t = 0$ and that $\mathscr{C}$ satisfies the Klein–Gordon equation. So, since $\mathscr{C} = \partial_0\mathscr{C} = 0$ at $t = 0$ and $\mathscr{C}$ satisfies the Klein–Gordon equation, it follows that $\mathscr{C}$ is identically zero, whence L_0 is also identically zero, from (A.5). QED. Note further that $L_\mu = 0$ is also a consequence of the argument of this paragraph. Thus the field equations (4) and (5), supplemented by the constraints $L_0 = \mathscr{C} = 0$ at $t = 0$, are equivalent to (A.2).

From this it is concluded that the above theory of a massive spin-one vector field in external potentials is causal, in the sense of Velo and Zwanziger, if and only if the above theory in the Stückelberg formalism is.

Finally, it is interesting to note the connection between the Lee–Yang determinants in these two theories. Noting that, in the vector formalism, the Lee–Yang determinant is the determinant of the symmetric matrix of the coefficients of the terms, in $\mathscr{L}(x)$, quadratic in $\sqrt{\tfrac{1}{2}}(\partial_0 V_k(x) - \partial_k V_0(x))$ and $\sqrt{\tfrac{1}{2}}mV_0(x)$, and using (A.1), (A.3), (12) and (16) it is readily seen that it is given by $\mathscr{D}(\eta)/-\eta^2$.

Thus the above quantum theory in the Stückelberg formalism is Lorentz invariant if and only if the corresponding theory in the vector formalism is.

Appendix 2

Lemma: If, for the Stückelberg field in external potentials, the characteristic determinant, $D(n)$, has a spacelike root for some values of the major coupling constant, then there are other values for which it has a timelike root.

Proof: The characteristic determinant may be written in the form

$$D(n) = \left| \begin{pmatrix} n^2 g_{\mu\nu} & 0 \\ 0 & n^2 \end{pmatrix} + \begin{pmatrix} \alpha_{\mu\nu} & \beta_\mu \\ \gamma_\nu & \delta \end{pmatrix} \right|$$

with $\alpha_{\mu\nu}, \beta_\mu, \gamma_\nu, \delta$ depending on n_μ, the field and external potentials, and all being proportional to g. Expanding $D(n)$ as the determinant of the sum of two matrices gives the following form:

$$D(n) = \sum_{r=0}^{5} a_r (n^2)^r$$

with a_r dependent on $\alpha_{\mu\nu}$, β_μ, γ_ν, δ in such a way that changing the sign of g sends a_r into $(-1)^{5-r} a_r$.

Now if $n^2 = f(a_r)$, say, gives (implicitly, since a_r depends in general on n_μ) a spacelike root of $D(n)$, for some value of g, then, since changing the sign of g sends $D(n)$ into

$$-\sum_{r=0}^{5} a_r(-n^2)^r .$$

it follows that $-n^2 = f(a_r)$ gives a timelike root of $D(n)$, for the value of g with opposite sign. QED.

References

Capri A Z 1969 *J. math. Phys.* **10** 575–80

Charap J M 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 393–408

Dowker J S and Mayes I W 1971 *Nucl. Phys.* B **29** 259–68

Jenkins J D 1973a *J. Phys.* A: *Math., Nucl. Gen.* **6** 1211–6

—— 1973b *Lett. Nuovo Cim.* **7** 559–62

Kvitky J S and Mouton J O 1972 *Prog. theor. Phys.* **48** 1693–707

Lee T D and Yang C N 1962 *Phys. Rev.* **128** 885–98

Mathews P M and Seetharaman M 1973 University of Texas (Austin) preprint

Matthews P T 1949 *Phys. Rev.* **76** 684–5

Schroer B, Seiler R and Swieca J A 1970 *Phys. Rev.* D **2** 2927–37

Stückelberg E C G 1938 *Helv. Phys. Acta* **11** 299–328

Suzuki T and Hattori C 1972 *Prog. theor. Phys.* **47** 1722–42

Takahashi Y 1969 *An Introduction to Field Quantization* (London: Pergamon)

Velo G and Zwanziger D 1969a *Phys. Rev.* **186** 1337–41

—— 1969b *Phys. Rev.* **188** 2218–22

—— 1971 *Lectures from the Coral Gables Conference on Fundamental Interactions at High Energy* vol 4, ed M Dal Cin, G J Iverson and A Perlmutter (New York: Gordon and Breach)